Geometric-Optical Modeling of a Conifer Forest Canopy

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Abstract—A geometric-optical forest canopy model that treats conifers as cones casting shadows on a contrasting background can explain the major portion of the variance in a remotely sensed image of a forest stand. The model is driven by interpixel variance generated from three sources: 1) the number of crowns in the pixel; 2) the size of individual crowns; and 3) overlapping of crowns and shadows. The model uses parallel-ray geometry to describe the illumination of a three-dimensional cone and the shadow it casts. Cones are assumed to be randomly placed and may overlap freely. Cone size (height) is distributed lognormally, and cone form, described by the apex angle of the cone, is fixed in the model but allowed to vary in its application.

The model can also be inverted to provide estimates of the size, shape, and spacing of the conifers as cones using remote imagery and a minimum of ground measurements. Field tests using both 10- and 80-m multispectral imagery of two test conifer stands in northeastern California produced reasonable estimates for these parameters. The model appears to be sufficiently general and robust for application to other geometric shapes and mixtures of simple shapes. Thus it has wide potential use not only in remote sensing of vegetation, but also in other remote sensing situations in which discrete objects are imaged at resolutions sufficiently coarse that they cannot be resolved individually.

Keywords—Plant canopy model, remote sensing, forest reflectance modeling, geometric probability

I. INTRODUCTION

MATHEMATICAL modeling of plant canopies is a research field that has been highly active in recent years. With the Duntley equations as a basis, many models have been developed for optical wavelengths. They are parameterized by such variables as \( \rho \) and \( \tau \) (reflectance and transmittance of the leaf, respectively), leaf area, and the leaf angle distribution. Most of these models are one-dimensional, in that the canopies vary only with height above the soil surface.

During the past three years, the authors have pursued the development of a plant canopy model that adopts a different perspective, treating the plant canopy as an assemblage of large solid three-dimensional objects. Utilizing optical principles and parallel-ray geometry, we have modeled a forest as a collection of randomly located cones that are illuminated at an angle and cast shadows on a background. We have also approached the modeling problem with the idea that the canopy will be imaged by a remote-sensing device, and, further, that the resolution of such a device will be sufficiently fine that the geometry of the pixel will interact with the size and placement of the cones. (In other words, a forest imaged by multispectral scanner in which the pixel size is several times greater than average size of the trees.) This approach therefore implies that a reflectance model will have to predict not only the average reflectance of the canopy, but also the variance in reflectance from pixel to pixel.

Another concern has been to try and maintain invertibility in our model, either directly through the use of appropriate side constraints or indirectly through iterative parametric estimation techniques. In this way, we have developed procedures to estimate the parameters of size, shape, and spacing which drive the model from the reflectance values that are observed. Although invertible models are usually more difficult to formulate, the development of image-based models has allowed us to exploit interpixel variance as an information source for inversion. Our emphasis on inversion, which is not a characteristic of most existing canopy models, is also somewhat unique.

It is important to understand that our model is based more strongly on geometric probability than the physical laws of radiative transfer. As input, the model requires specifying the size, shape, and density of the cones, the size of the pixel, the angle of illumination, and the relative brightness of the cones and their background under conditions of both shadowing and direct illumination. The output is the average brightness of a pixel and its variance. In inversion, the inputs are the pixel brightness values, along with parameters describing the angle of illumination and the reference brightness values of the illuminated and shadowed cones and background. The solution estimates the mean height, apex angle, and density of the cones.

This paper fully describes our geometric-optical canopy model, including as well some general derivations of geometric probability that are required along the way. It also summarizes our experience in applying the model to real images of forests and testing its invertibility in two forests stands for which the driving parameters of size, shape, and spacing are known. It concludes by analyzing the impact on the inversion procedure of some of the assumptions that are made in constructing the model and commenting on the applicability of our methods to dense forest canopies and fine-resolution imagery.

II. PREVIOUS WORK

A. Mathematical Plant Canopy Modeling

In the past decade, many mathematical plant canopy models have been devised. Space does not permit a full review of these models here; instead, the reader is referred to Smith and Ransom [1], Smith [2], and Strahler
et al. [3], as well as other papers in this issue. A number of models, however, have been developed that utilize a geometric-optical approach, and are thus relevant to the present discussion. Examples are the geometric-optical models developed by Richardson et al. [4] and Jackson et al. [5] for row crops. In addition, several models combine both radiative transfer and geometric optics approaches. These include the extended versions of the one-dimensional Suits model [6] for row crops developed by Verhoef and Bunnik [7] as well as Suits [8], and the three-dimensional reflectance model of Norman and Welles [9] based on a rectangular array of ellipsoids simulating crop plants.

1) Three-Dimensional Models: The geometric-optical model for a conifer forest canopy presented here relies on the three-dimensional nature of the forest canopy. Only a few models have been developed for the reflectance of three-dimensional surfaces. Kimes and Kirchner [10] developed a general modeling framework for a heterogeneous scene that utilizes a three-dimensional radiative transfer model. Colwell’s three-dimensional model of desert vegetation [11] utilized bare soil, erect stems, and their shadows as components to determine the coverage of plants by observing Landsat reflectances at two different sun angles. Kimes [12] developed a thermal model for row crops as repeating extended rectangular solids with three geometric parameters—width, height, and spacing between rows. By measuring thermal emission at different viewing angles, Kimes inverted his model to obtain crop geometry.

The modeling effort that is perhaps most directly relevant to our three-dimensional canopy model is that of Egbert [13, 14], who modeled optical bidirectional reflectance from shadowing parameters of surface projections or perturbations. More recently, Otterman [15, 16] has developed a model for a surface covered with vertical cylindrical perturbations. Like Egbert’s model, Otterman’s model assumes that the protrusions are small and numerous. Neither model is directly applicable to open forest canopies, in which the perturbations (trees) can occur in low densities and/or are relatively large with respect to the size of the resolution cell.

2) Scene Component Models: Another feature of our geometric-optical canopy model is that it treats total reflectance as a linear composite of scene components as weighted by their relative areas within the scene. Heimes and Smith [17] used this approach to investigate the variability in reflectance of two forest sites in mountainous terrain. By explicitly modeling the contribution of a shadow component, their study recognized the importance of shadows in determining the spectral response of a threedimensional scene.

B. Height and Spacing Functions for Conifer Stands

1) Spacing Functions: The conifer forest canopy model presented here requires specifying the height and spacing functions for trees within the stand. The spatial pattern of plants has been a topic of theoretical interest to ecologists and botanists for many years [18], [19]. Nearly all such studies have relied in a Poisson model—solid plant locations are equilike and independent. Although our previous work showed that a two-parameter Neyman type A model [20] better fit counts of ponderosa/Jeffrey pines in 80-m quadrats, our field work in the specific target stands used for field verification has shown a reasonable fit to the Poisson model for intermediate and dense stands at pixel (quadrat) sizes of 10–30 m.

2) Height Distribution: A normal distribution is perhaps the simplest choice to parameterize a random variable. The simple normal model, however, is not adequate to explain the variation of tree heights in forest stands. Instead, relatively simple but effective model, the lognormal distribution, has been widely applied [21, 22]. Our field work has also shown a good fit for a lognormal model in most of the test stands we have examined. Accordingly, a lognormal height distribution is used in our model development.

III. VARIABLES AND ASSUMPTIONS USED IN THIS PAPER

A. Context

Unlike many of the canopy methods described earlier, our model was developed specifically within the context of the remotely sensed digital image. In this context, the measurements of exiting radiation available to calibrate, verify, or invert a model are obtained from a digital image such as that produced by Landsat’s multispectral scanner (MSS) or thematic mapper (TM) instruments. The individual pixels of the image from an area of homogeneous forest canopy can thus be taken as replicate measurements of reflectance. The digital-image context also implies that the scene is illuminated at a constant angle that is known a priori; a further assumption is that the sensor will be nadir or nearly nadir looking. Atmospheric effects are also neglected, in that the signatures needed to characterize the components within the scene are assumed to be separable, distinct, and constant in the face of differential absorption, backscattering, etc. In addition, our modeling effort will be restricted to flat sites. This restriction means that we can ignore changes in geometry that occur with slope, since trees grow upright no matter what the inclination of the ground surface. Some of these assumptions or restrictions will be relaxed later; however, they provide a context that conforms well to Earth applications of remote sensing.

B. Assumptions

The fundamental assumption underlying our geometric-optical model is that conifer forest stands can be modeled geometrically as arrays of cones casting shadows on a contrasting background. The cones are of uniform shape, parameterized by a single constant—the apex angle of the cone—although this assumption will be relaxed somewhat later. The heights of the cones are lognormally distributed. Although the mean height is not known, we will assume that the coefficient of variation (ratio of standard
deviation to mean) for the log height values has been determined. We will further assume that trees (cones) are located randomly within and between pixels—that is, the counts of trees from pixel to pixel varies as a Poisson function, depending on the size of the pixel and density of the trees. In general, overlapping of crowns and shadows is permitted. A final assumption, discussed in more detail earlier, is that the reflectance of a pixel can be modeled as a sum of the reflectances of its individual scene components as weighted by their respective areas within the pixel. Taken together, these assumptions imply that pixel-to-pixel variance in reflectance will arise from three sources: 1) variation in the number of trees (cones) from pixel to pixel; 2) variation in the size (height) of trees both within and between pixels; and 3) chance variation in overlapping of crowns and shadows within the pixel.

C. Variables and Notation

1) Variables Associated with Tree Crowns:

\( \alpha \) One-half of the crown apex angle. Assumed constant within the stand. This assumption is relaxed later.

\( r \) Radius of crown as cone at the base. Lognormally distributed.

\( h \) Crown height. Lognormally distributed. Since \( \alpha \) is constant, \( h = r \cot \alpha \).

\( C_h \) Coefficient of variation (ratio of standard deviation to mean) for heights.

\( C_r \) Coefficient of variation of radius. If \( \alpha \) is fixed, then \( C_h = C_r \) since \( r \) is a linear function of \( h \).

2) Variables Associated with a Pixel:

\( A \) Pixel size. Usually taken as having a unit area.

\( n \) Number of trees (cones) in a pixel. Distributed as a Poisson; independent of other variables.

\( R^2 \) Average of squared radii within the pixel. That is

\[
R^2 = \frac{1}{n} \sum_{i=1}^{n} r_i^2, \quad i = 1, \ldots, n.
\]

\( m \) Ratio of sum of squared cone radii to area of pixel. This is

\[
m = \frac{nR^2}{A} = \sum_{i=1}^{n} r_i^2/A.
\]

Dimensionless.

A problem arises with the boundary of the pixel—what happens to a portion of the crown or its shadow that passes out of the pixel? Here we assume that the pixel will be replicated in all directions (i.e., will be surrounded with itself), and thus any excluded area on one side of the pixel will be included on the opposite side. This assumption will lead to a model that underestimates within-pixel variance slightly, but unless the pixel size is close to the tree size, the effect should not be large.

3) Variables Associated with the Timber Stand:

\( N \) Mean of \( n \) for all pixels. For the fully random model, this is the value of the Poisson parameter.

\( C_d \) Dispersion coefficient (variance-to-mean ratio) of \( n \). That is, \( C_d = V(n)/n \). If \( n \) is distributed as a Poisson function, \( C_d \approx 1 \). If not, \( C_d \) will depend on the pixel size \( A \). For the clumped or patchy distributions that characterize large quadrats in natural forests, \( C_d \) will increase with \( A \).

\( H \) Population mean of \( h \).

\( E(r) \) Population mean of \( r \).

\( V(r) \) Population variance of \( r \). \( V(r) = C_r^2 (E(r))^2 \).

\( E(r^2) \) Population mean of \( r^2 \).

\( V(r^2) \) Population variance of \( r^2 \).

If \( r \) is lognormally distributed, then \( r^2 \) is also lognormally distributed. We can then show from the definitions of \( E \) and \( V \) that

\[
E(r^2) = (1 + C_r^2) E(r)^2
\]

and

\[
V(r^2) = w(E(r^2))^2
\]

where

\[
w = (1 + C_r^4)^2 - 1.
\]

\( R^2 \) Mean value of \( R^2 \) for all pixels. That is, \( E(R^2) \).

\( R \) The square root of \( R^2 \), i.e., \( \sqrt{E(R^2)} \).

\( V(R^2) \) Variance of \( R^2 \).

If \( n \) is a constant and \( r \) is randomly distributed in the spatial domain, then \( R^2 \approx E(r^2) \). Also, \( R^2 \) is a sample mean, and thus \( V(R^2) = V(r^2)/n \).

\( M \) Mean of \( m \) for all pixels in the stand.

\( V(m) \) Variance of \( m \).

IV. Nonoverlapping Variance-Dependent Model

As the first stage of our modeling efforts, we developed a simple mathematical model that applies to sparsely forested stands when overlapping effects between coniferous crowns and/or shadows can be neglected [23], [20]. This model is invertible, thus allowing the direct calculation of height and spacing of trees from remotely sensed reflectance values. Inversion of the model, however, requires calculation of interpixel variance over a timber stand. Since the model requires interpixel variance to be known, we refer to it as the "variance-dependent" model.

A. Geometry of Model

Fig. 1 shows the geometry of a cone illuminated at a zenith angle \( \theta \). The apex angle of the cone is \( 2\alpha \). If \( \theta > \alpha \), a shadow will be cast. The right-hand diagram in the figure shows the projection of the cone and shadow onto the horizontal plane (i.e., the cone is "flattened"). The angle \( \gamma \) identifies the portion of the cone illuminated beyond the cone half. It is relatively easy to show that
The components: \( \text{sis} \) (i.e., \( \text{metric} \) \( \text{apex} \) \( \text{shadow} \) \( \text{tors.} \)).

or \( \text{mensional} \) \( \text{Ag} \) \( \text{AC} \) \( \text{Z} \) \( \text{2} \) \( \text{Kc} \) \( \text{Kg} \) \( \text{C} \) reveals \( \text{Reflectance} \) \( \text{7r/2} \) - \( \text{as} \) \( \text{Areas} \) \( \text{stated} \) \( \text{Fig.} \). \text{Reflectance} \( \text{Reflectance} \) \( \text{Reflectance} \) \( \text{Reflectance} \) \( \text{area-weighted} \) \( \text{constant).} \) \( \text{constant).} \) \( \text{constant).} \) \( \text{constant).} \)

\( \text{Illuminated} \) \( \text{shadowed} \) \( \text{Flat cone} \) \( \text{i.e.,} \) \( \text{area of green canopy projected to the sensor} \) \( \text{(pi/2 + gamma)} \) \( \text{r^2}; \) \( \text{shadowed area of the flat cone is} \) \( \text{(pi/2 - gamma)} \) \( \text{r^2}; \) \( \text{and the shadow on the ground has area} \) \( \text{(cot gamma - pi/2 + gamma)} \) \( \text{r^2}. \) \( \text{Summation of these areas is} \) \( \text{(cot gamma + gamma + pi/2)} \) \( \text{r^2}. \) \( \text{We shall denote the quantity} \) \( \text{(cot gamma + gamma + pi/2)} \) \( \text{as} \) \( \Gamma; \) \( \text{it can be thought of as a dimensionless geometric form parameter associated with the cone} \) \( \text{and its shadow that is dependent on the illumination angle} \) \( \text{and the apex angle of the cone.} \)

B. Reflectance of an Individual Pixel

As stated earlier, we model the reflectance of the pixel as an area-weighted sum of the reflectances of the four spectral scene components. New terms will need to be defined.

1) Reflectance Vectors: Multispectral reflectance vectors. These can also be thought of as points in multidimensional feature space.

\( \text{G} \) \( \text{Reflectance vector for a unit area of illuminated background (constant).} \)

\( \text{C} \) \( \text{Reflectance of a unit area of illuminated crown (constant).} \)

\( \text{Z} \) \( \text{Reflectance of a unit area of shadowed background (constant).} \)

\( \text{T} \) \( \text{Reflectance of a unit area of shadowed crown (constant).} \)

\( \text{S} \) \( \text{Reflectance of a pixel. Variable; depends on number and size of cones in pixel.} \)

2) Areas and Proportions: Variables describing areas or proportions for scene components.

\( \text{A_s} \) \( \text{Area of illuminated background within the pixel.} \)

\( \text{Quantity} \) \( \text{(A - A_s)} \) \( \text{is termed "covered area" in text later.} \)

\( \text{A_c} \) \( \text{Area of illuminated crown within a pixel.} \)

\( \text{A_z} \) \( \text{Area of shadowed background within a pixel.} \)

\( \text{A_l} \) \( \text{Area of shadowed crown within a pixel.} \)

\( \text{K_s = A_s/A. Proportion of pixel not covered by crown or shadow.} \)

\( \text{K_c = A_c/(A - A_s). Proportion of area covered by crown and shadow that is in illuminated crown.} \)

\( \text{K_l = A_l/(A - A_s). Proportion of covered area in shadowed crown.} \)

\( \text{K_z = A_z/(A - A_s). Proportion of covered area in shadowed background.} \)

3) Geometric Relationships: From the geometry of the cone model, we have the following simple relations:

\[ (A - A_s) = \sum_i r_i^2 \Gamma = \text{Am} \Gamma \]

\[ (A - A_s) = A_c + A_l + A_z \]

\[ K_s = 1 - m \Gamma \]

\[ K_c = (\pi/2 + \gamma) / \Gamma \]

\[ K_z = (\Gamma - \pi) / \Gamma \]

\[ 1 = K_c + K_z + K_s. \]

4) Modeling the Reflectance: The average reflectance of a pixel can then be written as a linear combination of four components.

\[ S = (A_g \cdot G + A_c \cdot C + A_z \cdot Z + A_l \cdot T) / A. \]

Substituting expressions from above into this equation

\[ S = K_s \cdot G + (1 - K_s) \cdot (K_c \cdot C + K_z \cdot Z + K_l \cdot T). \]  (1)

Since \( K_c, K_z \), and \( K_l \) sum to one, the expression \( (K_c \cdot C + K_z \cdot Z + K_l \cdot T) \) represents a point in multispectral feature space lying within a triangle with vertices at \( C, Z, \) and \( T \). Its position is dependent on \( K_c, K_z, \) and \( K_l \), which in turn are simple functions of the scene’s geometry (apex angle \( \alpha \) and illumination angle \( \theta \)). We will refer to this point as \( X \); it can be thought of as the average reflectance of a cone and its associated shadow. The only variable in the right side of (1) is thus \( K_s \), which is a linear function of \( m \). When \( m \) varies, \( S \) will vary along a straight line connecting points \( G \) and \( X \).

Substituting the geometric expressions earlier for \( K_c, K_z, \) and \( K_l \) into (1) yields

\[ S = G - Gm \Gamma + Xm \Gamma. \]

Rearranging, we have

\[ m \Gamma (G - X) = (G - S). \]  (2)

In the last expression, \( G - S \) and \( G - X \) are vector differences; however, \( G - S \) lies on the line \( G - X \) and therefore the equation is actually scalar. Using the notation \( |GS| \) to indicate the length of the vector connecting \( G \) and \( S \), we have

\[ m = \frac{|GS|}{\Gamma |GX|}. \]  (3)

Although there is thus in theory a unique solution for \( m \), noise will always be present in \( S, \alpha, \) and the component
signatures that determine $X$. Therefore, the position of $S$ will deviate from the line segment $GX$. Under these conditions, (2) is no longer a scalar relationship but a set of linear equations in which $m$ is overdetermined.

One solution to this problem is to use a maximum likelihood approach for the best fit of $m$ to the reflectance signatures [24]. The approach we use here, however, selects a projection in feature space that maximizes the signal-to-noise ratio. In the absence of noise, (1) becomes a scalar equation in that projection and thus possesses an exact solution. Note by taking the partial derivative of (2) for $S$, $G$, and $C$ that the error in $m$ will be proportional to $1/|GX|$. Thus a projection for which the line segment $|GX|$ has the greatest length will have the least sensitivity to error. In other words, we choose the band combination in which the background is spectrally most different from the average reflectance of the cone and its shadow.

C. Inverting the Model using the Variance of $m$

Assume that a timber stand consists of $K$ pixels, $i = 1, \cdots, K$. From (2), we can obtain a value of $m$ for each pixel. Then, the values of $m$ will have a mean and variance within the timber stand

$$M = \frac{1}{K} \sum_{i=1}^{K} n_i R_i^2;$$

$$V(m) = \frac{1}{K} \sum_{i=1}^{K} (n_i R_i^2 - M)^2.$$ 

Let us now assume that height (and thus $r$) is independent of density. Thus expressions for the mean and variance of independent products will apply

$$M = E(nR^2) = E(n) \cdot E(R^2) = NR^2 \quad (4)$$

and

$$V(m) = V(nR^2) = (R^2)^2 V(n) + N^2 V(R^2) + V(n) \cdot V(R^2). \quad (5)$$

Since $n$ is a Poisson function

$$V(n) = N. \quad (6)$$

Further

$$V(R^2) = V(r^2)/n \approx V(r^2)/N = w(E(r^2))^2/N. \quad (7)$$

Substituting (6) and (7) into (5), we finally obtain

$$V(m) = (N + wN + w)(R^2)^2$$

$$= (M + wM + wR^2) R^2. \quad (8)$$

In order to derive (8), $R^2$ and $V(R^2)$, which are parametric terms, are approximated using the sample mean and variance of $r^2$. Small errors are introduced by these approximations, but they may be ignored for our purposes. Solving (8) for $R^2$, we obtain

$$R^2 = \frac{(1 + w)^2 M^2 + 4 V(m) w}{2w} - (1 + w) M. \quad (9)$$

Thus given sample estimates of the mean and variance of $M$ determined from the reflectances of pixels in the stand, we can solve for $R^2$, and then for $N$, yielding the average size and density of trees in the stand.

The assumption underlying the use of the sample variance of $r^2$ as $V(R^2)$ is that each pixel is an independent sample of values of $r^2$. Other approximations can be also applied to (5). For example, if the interpixel variation of $r^2$ is more significant than intrapixel variation, we may use $V(R^2)$ directly as an approximation of $V(r^2)$. Then (8) becomes

$$V(m) = (1 + w) M R^2 + w M^2$$

and we obtain

$$R^2 = \frac{V(m) - w M^2}{(1 + w) M}. \quad (10)$$

Also, if the dispersion coefficient of $n$ is significantly different from 1, we may use $V(n) = NC_d$. Then (9) becomes

$$R^2 = \frac{((C_d + w)^2 M^2 + 4 V(m) w C_d)^{1/2} - (C_d + w) M}{2w C_d}. \quad (11)$$

The choices basically depend upon what a priori information we have.

No matter which approximation is used, however, the underlying expressions (4) and (5) will always yield relationships such that for a given $M$, the larger the $V(m)$, the larger is $R^2$. To make this point clearer, let us further apply the approximation formula $\sqrt{1 + x} \approx 1 + x/2$ to (9). We obtain

$$R^2 \approx \frac{V(m)}{(1 + w) M}. \quad (12)$$

Although (10) may not be sufficiently accurate to apply when $V(m)$ is fairly small, the expression shows that for a given $m$, the larger is $R^2$, the higher will be the variance of $m$—and since $S$ is a linear function of $m$, the “rougher” the surface will appear. This is obviously true for both manual interpretation of aerial photographs and digital processing of remotely sensed imagery.

Although the variance-dependent model has performed well in earlier work [20], it is only applicable to cases of low and moderate stocking. Thus we turned to a fuller consideration of how the reflectance of pixels is influenced by the variance in overlapping of objects within them.

V. OVERLAPPING MODEL

The problem of two-dimensional objects overlapping in a spatial field has been examined by both geographers and mathematicians. There are two ways to deal with this problem. The first is to assume that the centers of objects are randomly distributed at an average density over an infinitely large region. In 1971, Getis and Jackson [25] applied a Poisson model for the mean area polluted by randomly scattered pollution sources. In their model, a point is not polluted if it is not contained within the area associated with a pollution source. Thus the mean area not
polluted by any of $N$ sources is
\[ A_\text{g} = A e^{-N(a/A)} \]  
(11)
where $A$ is the overall area, $A_\text{g}$ is the unpolluted area, and $a$ is the area polluted by one source. More recently, Serra [26] proves a similar relation and a formula for the probability that neither of two points is covered by the objects for any multidimensional Poisson process locating points or volumes. Although this model predicts the mean, it does not account for the pixel-to-pixel variance.

Another approach is to assume a specific number of objects are distributed within a specific area, and then to assume this number varies as a Poisson or other function. In 1947, Garwood [27] proved that the mean proportion of undamaged area in a building or factory complex of size $A$ is
\[ E(A_\text{g}/A) = (1 - a/B)^k \]  
(12)
where $B$ is a finite target area which contains $A$; $k$ is the number of bombs which fall randomly within $B$; and $a$ is the area damaged by a single bomb. He also obtained the second order moment of $A_\text{g}/A$
\[ E((A_\text{g}/A)^2) = \int_A \left( \frac{B - 2a + q(x, y)}{B} \right)^k \, dx \, dy/A \]  
(13)
where $q(x, y)$ is the common area which is damaged by two bombs falling at $(0, 0)$ and $(x, y)$. Garwood’s result is directly relevant to the problem of cones with shadows falling randomly in a square or rectangular pixel.

In Garwood’s formulation, $B$ is required to eliminate difficulties at the boundary. An alternative method is to assume, as stated previously, that the pixel is replicated in all directions. This assumption also allows the use of Fourier transforms and theorems. We have obtained expressions for the first- and second-order moments of $A_\text{g}/A$ [28] that are quite similar to the results of Garwood
\[ E(A_\text{g}/A) = (1 - a/A)^k \]  
\[ E((A_\text{g}/A)^2) = \int_A \left( \frac{A - 2a + q(x, y)}{A} \right)^k \, dx \, dy/A \]  
(14)
to yield
\[ V(K_g) = \int_A \left( \frac{A - 2a + q(x, y)}{A} \right)^k \cdot dx \, dy/A - (1 - a/A)^{2k}. \]  
(15)
Thus (14) and (15) present relatively simple formulas for the mean and variance of $A_\text{g}/A$. These expressions conform with Ailam’s more general result for the moments of coverage and coverage spaces of randomly distributed objects [29]. The formulas are also validated by the good agreement observed between calculated values and results simulated by Monte Carlo modeling, as discussed in a later section. The formula for the mean agrees with the Getis–Jackson model (11) since $e^{-Na/A}$ is a very good approximation of $(1 - a/A)^N$ when $N$ is large and $a/A$ is small. (This can be seen easily by expanding both expressions into a Taylor series.) When $a/A$ is somewhat larger, our simulations show better agreement with $(1 - a/A)^N$, but $e^{-Na/A}$ will still be adequate for the purposes of this paper. Our general expression for the variance (15) also conforms with the specific result of Solomon [30], who employed a theorem of Robbins (1944, cited in [30]) to derive a recursive integral equation for calculating the mean and variance of coverage of random caps on a sphere. Other authors [31] have also applied Robbins’ theorem to different specific coverage problems with similar results.

We should emphasize that (15) is a very general result. Because $(1 - a/A)^2$ can be expressed in terms of $q(x, y)$, the variance in $A_g$ is a simple function of the autocorrelation function $q(x, y)$ and the number of objects. Although (15) refers to variance in the uncovered area, it also describes the variance of the covered area as well since the two areas are complementary. Thus it will apply to an object of any shape, as long as we can describe its autocorrelation function. When the sizes and shapes of the objects within the pixel are not the same, (14) and (15) become
\[ E(K_g) = \prod_{i=1}^{N} (1 - a_i/A) \]  
(16)
and
\[ V(K_g) = \int_A \left( \frac{A - 2a + q_i(x, y)}{A} \right)^k \cdot dx \, dy/A - \prod_{i=1}^{N} (1 - a_i/A)^2. \]  
(17)
In the first expression, $e^{-Na/A}$ can be used as a good approximation of $E(K_g)$ if the $a_i$ are small with respect to $A$. Thus $E(K_g)$ can easily be calculated for mixtures of simple shapes. In the second expression, the variance is a function of the products of the individual autocorrelation functions for each shape. Note that the cross correlation function for one shape with another need not be determined. In practice, this means that such scenes as mixtures of conical conifers and hemispherical deciduous trees can be explicitly modeled. In fact, it should be relatively easy to model any scene that can be described as a mixture of randomly distributed simple shapes. Thus our results have wide implications beyond the modeling of forest canopy reflectance.

VI. MONTE CARLO SIMULATION

In order to confirm the mathematical proofs of the preceding section, as well as to explore various other aspects of geometric modeling of forests, a Monte Carlo computer model was prepared to simulate pixels composed of cone-shaped tree crowns. In the simulation, the pixel is modeled as a two-dimensional array of many small subpixels, each of which is assigned a cover type (e.g., bright crown, shadowed crown, bright background, shadowed background). Much of our work used Landsat-sized pixels,
each 80 m by 80 m, with 1-m subpixels. SPOT-sized pixels were also modeled; these were 10 m by 10 m, with a subpixel size of approximately $\frac{1}{3}$ m.

The computer program carrying out the simulation is exercised in several steps. First, tree counts for as many pixels as are to be simulated are selected from either of three distributions: uniform (i.e., constant), Poisson, or Neyman Type A. Given that the number of trees to be located within the pixel has thus been determined, the next step is to specify the position of each. To locate each tree, a subpixel is chosen at random as its center. If a pure Poisson model has been chosen, trees may overlap fully and no restrictions are placed on the location of a center. However, we have also implemented a "hard-core" model in which the center of a tree may not be located within the crown of another tree. In this case, the crowns may touch and intersect only to a limited degree. The purpose of this option is to simulate situations in which intertree competition may be important in the location process.

Next, the program selects the height of each tree from a lognormal distribution using a mean and variance that are specified as input parameters. Since the apex angle of the cone and the illumination angle relative to the zenith are also specified, the program can then compute the geometry of the cone and its shadow and assign subpixels to sunlit and shadowed crown and background. In addition, the program also calculates the height of the center of each subpixel. This information is used to assign the appropriate cover type to each subpixel when shadows fall on crowns and crowns intersect. The height matrix is also used when a nonnadir viewing angle is simulated. In this case, the program calculates whether or not each subpixel is obscured. When the specified number of trees have been placed in the pixel, the program counts the subpixels in each type of cover and then goes on to the next simulated pixel. Statistics are accumulated as the program runs and summaries and tabulations are output at the end.

Fig. 2 presents an image of a simulated pixel generated by the Monte Carlo modeling program. The image is produced by assuming Lambertian reflectance from the curved surface of each cone. This 80 m by 80 m pixel uses a very fine subpixel mesh of approximately 0.3 m to yield a high-quality continuous-tone image. For this simulation, the solar zenith angle ($\theta$) is set at 30°, the apex angle of the cones ($2\alpha$) is 20°, the trees are uniformly 20 m high, and the pixel contains 80 cones. Viewing is from the zenith.

A. Modeling Bidirectional Reflectance

The bidirectional reflectance distribution function (BRDF) for this pixel was also calculated using the geometry of the scene and is shown in Fig. 3. Both the cone and background are assumed to be Lambertian surfaces with albedos for sunlit background, sunlit crown, and shadow of 0.7, 0.3, and 0.0, respectively. Fig. 3 presents the bidirectional reflectance plotted in a polar coordinate system in which $\varphi$ indicates the difference in azimuth angle between sun and sensor, and $\rho$ indicates the corresponding reflectance. The family of curves shows different sensor zenith angles, indicated in units of 10°.

The influence of the three-dimensional geometry on reflectance is clearly visible in the figure. At small azimuth differences, sun and sensor lie in nearly the same plane. The shadows can barely be seen and thus the scene appears brightest. As the viewing angle comes around and the sensor looks into the sun, shadowing becomes much more important and the reflectance decreases. The geometric effects also cause reflectance to decrease as the viewing angle becomes increasingly oblique. At such "low" viewing angles, the light background becomes increasingly obscured and thus the scene appears darker. The minor irregularities that produce spikes on the curves appear to be due to the chance placement of cones in this particular pixel (Fig. 2). For example, the spike at $\varphi = 170°$ is obviously explained by a gap in that direction, which reveals more sunlit background and produces a greater reflectance.

The shape of this reflectance function generally resembles that of Kriebel [32], and also exhibits features similar to those observed and modeled by Kimes et al. [40] for sparse crop canopies. These results emphasize the importance of the three-dimensional geometry of the canopy in determining its BRDF.

B. Simulations of Landsat-sized Pixels

The Monte Carlo model was first exercised to simulate 80-m pixels. The driving parameters for this simulation
were the same as those used in the BRDF simulation earlier, except that a coarser 1-m subpixel size was used. The count of trees per pixel was varied in steps of 10 from 1 to 200; with cones of 20-m height, this range spans canopy coverages from about 5 to 70 percent.

Fig. 4 plots \( 1 - K_g \), the area covered by crowns and shadows, as a function of \( m \Gamma \), which is the proportion of area covered by cones and shadows without overlapping. Also plotted are values calculated using \( 1 - e^{-m \Gamma} \); the curves are nearly coincident. The simulated and calculated variance of \( K_g \) (which is equal to the variance of \( 1 - K_g \)) is presented in Fig. 5. The two curves show reasonable agreement, although the simulated values seem slightly higher than the calculated values and depart from a smooth function due to random variation. The former effect may well be due to the fact that the same random number seed was used for each choice of \( m \Gamma \), which would cause some serial correlation in the results.

Fig. 6 presents the means of \( K_c, K_z, \) and \( K \) as a function of \( m \Gamma \). As the tree count increases, \( K_c \), the proportion of the covered area in shadowed background, decreases. This effect arises because progressively more of the shadows fall on surrounding trees as the density of trees increases. For the same reason, \( K_z \), the proportion of shadowed crown, increases. With \( K_z \) decreasing and \( K_c \) increasing, \( K \), the proportion of the covered area remaining in sunlit crown, remains nearly constant. Fig. 7 presents the variance of these three quantities as observed by simulation.

### C. Simulations of SPOT-sized Pixels

Because both Landsat data and an aircraft multispectral-scanner simulation of SPOT satellite data (discussed in a following section) were available for testing the model, SPOT-sized pixels were also simulated. For these runs, a 1-ft square subpixel size was used in a pixel measuring 33 ft by 33 ft—thus approximating a 10-m pixel. Although this subpixel grid, composed of 1089 cells, is coarser than the 6400-cell grid used in the Landsat pixel simulation, note that the objects are proportionately much larger. Thus these results should be comparable.

To match the characteristics of our SPOT simulator data and ground observations (described in more detail in a later section), we used a solar zenith angle \( \theta = 26^\circ \) and a cone apex half-angle of \( \alpha = 8.48^\circ \). Simulated and calculated values of means and variances for \( 1 - K_g \) are shown in Figs. 8 and 9. Again, the simulated and calculated curves show good agreement. The simulated means of \( K_c, K_z, \) and \( K \) are shown in Fig. 10. They show the same trends as Fig. 6. Note that they are presented as a function of \( m \pi \) (rather than \( m \Gamma \)), which is the ratio of the tree base area to the area of the pixel. The simulated variances for \( K_c, K_z, \) and \( K \) are shown in Fig. 11.

Because \( K_s \), the proportion of sunlit crown within the total covered area, remains relatively constant, its variance is much smaller than that of \( K_c \). Since \( K_z + K_s = 1 - K_c \), the variance of \( K_s + K_c \) is also small. Both \( K_s \) and...
$K_i$ are shadowed cover types and will have similar reflectances; thus we will be able to assume in later modeling that the overlap variances of $K_c, K_r$, and $K_z$ can be ignored.

VII. Modeling Spectral Reflectance of the Canopy

Given the spectral reflectances of the four scene components, it is possible to model the spectral reflectance of the whole pixel as a function of the height, density, and shape of the cones it contains. For this effort, we will consider the tree crown to be a medium-bright green reflector.

To provide a strong contrast, we will assume that the background is snow—a bright uniform spectral reflector in the visible and near infrared. Shadowed crown and shadowed background will be darker, but spectrally similar to their directly illuminated counterparts.

For convenience, we will work in a two-dimensional spectral space in which the reflectances are orthogonal and uncorrelated, corresponding approximately to a "greenness–brightness" transform [33]. This linear transform was derived from analysis of many Landsat images of agricultural scenes, which showed that the four spectral bands of Landsat data could be decomposed into two orthogonal axes of variation—one related to the density and depth of the plant canopy (greenness) and the other related to the brightness of the soil. From our viewpoint, the greenness, brightness transform represents the best two orthogonal axes separating tree crowns from background.

A. Coverage and Apex Trajectories

Fig. 12 presents an idealized plot of the four spectral components in a greenness–brightness feature space. The snow background ($G$) has the greatest brightness value, but as a fairly uniform spectral reflector it has low greenness. Illuminated crown ($C$) has the most intense greenness, and is also somewhat bright. Shadowed crown ($T$) is the darkest component of all, but it still exhibits some greenness. Shadowed background snow ($Z$) is lighter than shadowed crown, but rates a very low greenness. A pixel
without trees will be pure background, and thus will have reflectance \( G \). As trees are added to the pixel, the reflectance of the pixel \( S \) will move along a line toward the point \( X_o \), which represents the average reflectance of a single tree with its three spectral components. Overlapping, however, will occur, reducing the relative proportion of shadowed background \( K_s \) and increasing the relative proportion of shadowed crown \( K_c \). As a result, the path of the pixel as coverage increases will diverge away from \( X_o \).

As tree density goes to the limit, all the background will be obscured and \( K_c \) will become zero. The limiting reflectance \( X_\infty \) will then lie somewhere along the line \( TC \), with the exact position depending on the apex angle of the cone and the angles of illumination.

To model this situation mathematically, we first reconsider (1) for the proportions expressed as means

\[
S = (K_c C + K_s Z + K_t T)(1 - K_g) + K_g G. \tag{18}
\]

As before, the point \( X = K_c C + K_s Z + K_t T \) will represent the average reflectance of a cone and its associated shadow. Our analysis from the preceding sections, however, has yielded expressions for the average proportions. First, we know that

\[
K_g = e^{-m T}
\]

Then, since \( K_c \) is approximately a constant, and the mean area occupied by the bases of the cones is \( A(1 - e^{-m x}) \), we can show that

\[
K_c \approx e^{-m x}(\Gamma - \pi)/\Gamma
\]

and

\[
K_t \approx [(\pi /2 - \gamma) + (\Gamma - \pi)(1 - e^{-m x})]/\Gamma.
\]

By substituting these relationships into (18), the point \( X \) becomes a function of \( m \) and \( \alpha \). At \( m = 0 \)

\[
X_o = [(\pi/2 + \gamma) C + (\pi/2 - \gamma) T + (\Gamma - \pi)Z]\Gamma
\]

which is the average reflectance of a crown and shadow with no overlapping. At \( m = \infty \)

\[
X_\infty = T + (\pi/2 + \gamma)(C - T)
\]

which is the average reflectance of a shadowed crown with complete overlapping. Thus the “coverage trajectory” of \( S \) (Fig. 12) with increasing \( m \) will be determined by the apex angle. If \( m \) is fixed and \( \alpha \) is allowed to vary, an “apex trajectory” will result.

Fig. 13 presents families of coverage and apex trajectories for a reasonable range of values of \( m \) and \( \alpha \). (The values of \( C, Z, T \), and \( G \) on greenness–brightness axes are chosen from analysis of SPOT simulation data, discussed in a following section.) The trajectories are calculated explicitly from (18). The results of our simulation trails may also be plotted as coverage trajectories to verify the calculations.

B. Relevance to Tasseled Cap Transform

It has long been known that the spectral reflectance pattern of crop canopies in Landsat MSS bands is basically two-dimensional. The pattern can be represented by the so-called “tasseled cap” shape, in which the crop canopy becomes increasingly darker and greener as it progressively obscures the lighter soil beneath it. As the crop canopy reaches maturity, its reflectance trajectory changes direction, turning to become brighter, before losing greenness as the crop senesces. This same pattern has also been recognized for forests [34], and has been attributed to the overlapping of shadows onto tree canopies [11]. The geometric-optical model we derived earlier presents a deterministic explicit description of this effect, at least for forests. It arises from the progressive obscuring of the soil or understory background coupled with a reduction in shadowed area as high coverages are reached. Although they are not explicitly modeled here, similar geometric effects may well explain the “tasseled cap” shape for crops.

C. Numerical Solution of (18)

Our expression for the greenness and brightness of a pixel with overlapping (18) is essentially a set of two nonlinear simultaneous equations with two unknowns, \( m \) and \( \alpha \). Although it is difficult to obtain a closed-form solution for these equations due to the nonlinear terms \( e^{-m} \) and \( \Gamma \), values of \( m \) and \( \alpha \) can be found numerically without much difficulty. Because these values are estimated from the model, we will refer to them as \( m' \) and \( \alpha' \).

Since the signature of each pixel is modeled as a linear function of the signatures of the four components it contains, the solution of (18) exists for any \( S \) inside the polygon \( ZCIG \) (Fig. 12). When \( S \) falls outside of this polygon due to noise or other error, this situation can be recognized and the value can be omitted from further processing. There may also be some pixels with little or no canopy cover. The reflectance of these pixels should then be \( G \), in the ideal case. However, as a result of topographic variation, shadowing from other pixels, variation in surface composition, etc., these pixels will present a range of reflectance patterns analogous to the “soil axis” of Kauth and Thomas [33]. Some of these values will fall into the polygon \( ZTCG \) and (18) will give nonzero values of \( m' \) and \( \alpha' \) as solutions for them. If such values are not
close to \( Z \), the resulting \( m' \) will be very small, i.e., close to a correct solution. If the value lies near the segment \( ZT \), the resulting \( m' \) value may be large; however, the value \( \alpha' \) will be very small. These types of errors can be easily recognized and such pixels can be discarded.

When \( S \) values lie on or near the segment \( TC \), the solution of (18) will be unstable. In this case, both \( e^{-m} \) and \( e^{-m'} \) are nearly zero, and the sensitivity of \( m' \) to changes in \( S \) will be proportional to \( e^m \). For this reason, the solution of (18) will be increasingly subject to error as \( m \) becomes large. Thus there will probably be a practical upper bound on stand densities for which \( m' \) can be derived reliably.

Without noise or overlap variance, we could simply apply (3)–(8) to the \( m' \) values derived from (18) and invert the model directly. Noise and overlap variance, however, exist; further, the solution of (18) is sensitive to noise at large \( m \) values. Under these conditions, the mean and especially the variance of \( m' \) will depart from modeled values. In spite of these problems, it is still possible to invert the model and obtain estimates of size and spacing for the objects in the scene. To show this, let us describe the problem as follows.

Using a geometric-optical approach, we have modeled how the distributions of \( r^2 \) and \( n \) produce a distribution of \( S \) values within an image. Even though the procedure is somewhat complicated, the only unknown parameters are \( E(r^2) \) and \( N \) for the modeled procedure. We also have a sufficient number of observations of \( S \). The problem is then whether we can estimate the unknown parameters from observations of \( S \). The problem is a typical one of parametric point estimation, and many techniques, such as maximum-likelihood estimation, minimum \( \chi^2 \), least squares, etc., are available for its solution.

The analysis of (18) in the preceding section showed that noise in \( S \) values may prevent accurate inversion. Since some of the techniques cited earlier do not require that the observations cover the whole range of the distribution, some observations can be discarded if we know they are likely to contain error. In this application, these are most likely to be values that produce large values for \( m' \). An appropriate strategy is then to make the inversion of the model rely on only a limited range of the \( m \) distribution. This amounts to fitting a curve in an area of the curve for which the data are most accurate.

VIII. FIELD TESTS

A. Remotely Sensed Data and Field Measurements

To test the model on real data, we used 10-m simulated SPOT imagery, produced by a modified Daedalus scanner on a Lear jet aircraft as part of the 1983 SPOT simulation campaign. The image was acquired on June 25, 1983. Four bands of data were supplied: a 10-m panchromatic band (\( B4, 0.51–0.73 \mu m \) nominal), and three spectral bands in the green (\( B1, 0.50–0.59 \mu m \)), red (\( B2, 0.61–0.68 \mu m \)), and infrared (\( B3, 0.79–0.89 \mu m \)) at 20-m resolution. For our tests of model inversion, the multispectral bands were expanded to 10-m resolution by replicating pixels. Also used was a Landsat-4 MSS image from June 11, 1983. The original pixel values corresponding to a resolution cell size of 56 by 79 m were used. These data were processed using principal components into transformations that resembled brightness and greenness. The transformation for the SPOT data is

\[
\text{Brightness} = 0.28B1 + 0.43B2 + 0.75B3 + 0.41B4
\]

\[
\text{Greenness} = 0.65B3 - 0.40B1 - 0.40B2 - 0.51B4.
\]

The transformation for the Landsat data is

\[
\text{Brightness} = 0.50B1 + 0.51B2 + 0.54B3 + 0.45B4.
\]

\[
\text{Greenness} = 0.36B3 + 0.64B4 - 0.50B1 - 0.46B2.
\]

For coordinated ground study, two test areas were selected in the Goosenest Ranger District of the Klamath National Forest (northeastern California): a dense mature high-elevation red fir stand, and a more open mixed conifer stand, largely of ponderosa/Jeffrey pine with some white fir, at a lower elevation. Field measurements in each plot were taken in July, 1983; they are described more fully in [28].

Analysis of the field data showed the mixed conifer stand to be composed of ponderosa and/or Jeffrey pine (78 percent) and white fir (21 percent), with a canopy coverage of about 30 percent. For this plot, the background understory consisted of a mixture of grass, bare soil, and a few shrubs.

The red fir site was dominated by red fir (69 percent) with white fir (26 percent) and pine (< 5 percent) comprising the remaining components. Canopy coverage was about 80 percent. The understory was largely a litter layer of cones, needles, and branches at the time of the field work, but was covered by snow at the time the imagery was collected.

Further analysis of the data [28] showed that the tree counts within the pixels were fit well by a Poisson distribution at the small (10-m) pixel size. The dispersion coefficient \( C_d \), however, increased with pixel size, requiring specific correction at Landsat pixel size. Tree height was observed to fit a lognormal model well. A fuller description and analysis of tree data are planned for a future publication.

B. Inversion Procedure

To test the geometric-optical canopy model in the red fir and mixed conifer stands, we used the following procedure.

1. Using air photos, identify and delineate the two stands on the imagery.

2. Pooling pixels from the two sites, carry out a principal components axis rotation and project the data onto the first two axes. Figs. 14 and 15 plot the rotated SPOT simulator data for the red fir and mixed conifer stands, respectively.

3. Determine the rotated signatures of the four spectral components from the imagery by careful examination of values for individual pixels. The apex and coverage trajectories shown in Fig. 13 are plotted using these spectral
component signatures and are at the same scale as Figs. 14 and 15. A direct comparison of Figs. 14 and 15 with 13 shows that the red fir site has more overlapping and a smaller mean apex angle than the mixed conifer site. This observation conforms well with the field data collected.

4 Use (18) to determine $m'$ and $\alpha'$ for each pixel. (At present, this is implemented through a table look-up procedure.) Thus we transform each pair of greeneriness and brightness values to a pair of $m'$ and $\alpha'$ values. By projecting these pairs to two one-dimensional distributions, we obtain a “projected $m$ distribution” and a “projected $a$ distribution.” From the latter, we estimate the mean apex angle $\alpha$.

5 Begin an iterative procedure to find values of $N$ and $R^2$ that will generate an $m$ distribution matching that observed. Select starting values for $N$ and $R^2$ to generate an $m$ distribution. To reduce computing time, use a normal distribution with the mean and variance derived from (4) and (8). From the normal, generate a probability distribution $p(m)$ for values of $m$.

6 For every value of $m$, generate a normal distribution $p(K_s|m)$ using values for the mean and variance derived from (14) and (15). Also calculate values for $K_2$, $K_3$, and $K_4$, as discussed earlier, their variances can be ignored.

7 Now find the distribution of $K_s$ as

$$p(K_s) = \int p(m) \, p(K_s|m) \, dm.$$ 

From this, use (18) to generate an $S$ distribution that therefore characterizes the values chosen for $N$ and $R^2$.

8 Apply the lookup table based on (18) to the generated $S$ distribution, obtaining a “generated $m$ distribution” which contains overlap variance. Calculate the mean square error at a limited range of $m'$ between the generated and projected $m$ distributions.

9 Repeat the four preceding steps using different pairs of $N$ and $R^2$. Use the pair of $N$ and $R^2$ values which yields the least mean square error as the result.

10 Last, apply the relations $E(r) = R/\sqrt{(1 + C_r^2)}$ and $H = E(r) \cot (\alpha)$ to obtain mean height.

Since 80-m Landsat MSS data were also available for the two sites, the procedure was repeated using the Land- sat imagery as well.

C. Results

Table I presents the results for 10-m SPOT simulator data and compares them to values as measured in the field. In inversion of the model, $C_r$ is required; the value obtained by field measurement is used.

A close examination reveals that the calculated results do not reproduce the field measurements exactly, but differ from them in some respects. For the red fir site, the calculated average radius and apex angle are somewhat smaller than observed, and the height is significantly smaller. The discrepancy in average height may be explained in part by the field sampling procedures, which overestimated the mean height [28]. In spite of these differences, the contrast between the two sites is evident. The red fir site is much more densely covered than the mixed conifer site, and includes trees that have narrower crowns (i.e., red firs as compared to ponderosa/Jeffrey pines). Thus the inverted model clearly reveals the major structural differences between the two sites.

Table II presents the results of model inversion for

![Fig. 14. Plots of red fir pixels in brightness–greenness space (SPOT simulation).](image1)

![Fig. 15. Plots of mixed conifer pixels in brightness–greenness space (SPOT simulation).](image2)}
80-m MSS data. Again, there are some differences between observed and calculated results—the spacing parameter for the red fir stand is significantly underestimated, for example—but the contrasts between the two sites emerge well.

Inversion of the model for 80-m data required an additional piece of information—\(C_d\). Since earlier studies of spacing in similar conifer stands showed that this value differs from one at larger pixel sizes [35], we estimated this parameter from field data by regression of \(C_d\) values with pixel sizes in the range of 14.2 ft square to 56.7 ft square. Obviously, the 80-m Landsat pixel is larger, and thus our estimates may be subject to significant error from this source.

### IX. Discussion

#### A. Sensitivity to Component Signatures

An assumption that underlies our inversion procedure is that the signatures of the four spectral components are known accurately. Without field radiometric measurements, however, we are forced to estimate the signatures from the image itself. Thus it is reasonable to examine the sensitivity of the inversion procedure to error in determination of the component signatures.

An analysis of (18) reveals that error in a component signature will influence the signature of the pixel in direct proportion to the area within the pixel having that signature and in inverse proportion to the distances between the signature in error and the others. The first source of error will depend on the cover and illumination geometry for the shape; the second will depend on the spectral separability of the four components.

As a practical test of sensitivity, we modified the signatures for the SPOT data and reinverted the model for the red fir and mixed conifer sites. Due to limits in computer time, it was not possible to carry out a systematic sensitivity test of each parameter. The results are summarized in Tables III and IV. In general, the results of the inversion are more sensitive for the mixed conifer site than for the red fir site. This effect arises because the shrub-covered background at the mixed conifer site is less separable from trees and shadows than in the snow-covered background of the red fir site. The average background proportion (0.7) for the mixed conifer site is also larger, multiplying this effect.

The tables show the effects of changing the separability between some of the signatures by as much as 20 percent. The model responds with varying estimates of size, spacing, or apex angle by somewhat lower percentages. Yet in spite of such changes, the primary differences between the two sites remain obvious. Thus we conclude that the model is not overly sensitive to error in the determination of component signatures.

#### B. Field Calibration

The field calibration required to invert the canopy model includes determining \(C_a\) for SPOT data, and in addition, \(C_d\) for MSS. The inversion method used here could also be applied to the solution of a four-parameter model that includes \(C_a\) and \(C_d\) in addition to \(E(r)\) and \(N\) as unknowns (e.g., methods used by Goel [36]-[39]). This approach is possible because the \(m\) distribution is governed by these four parameters and it should be possible to estimate them from the observed \(m\) values. This, however, may be very difficult due to the existence of noise as well as the limitations on the number of pixels that can be sampled from a homogeneous stand. On the other hand, it may be possible to estimate \(C_d\) from multiresolution imagery, since \(C_d\) is proportional to the size of the pixel [35]. This approach remains to be developed. \(C_r\), however, will probably have to be established through field calibration, although it may be that sufficient forestry data will exist to estimate \(C_r\) at many locations without ground measurements.

#### C. Resolution

For inversion of the canopy reflectance model, the 10-m resolution of the simulated SPOT data is superior to the 80-m resolution of Landsat MSS. Since \(V(m)\) is inversely proportional to the pixel size, \(V(m)\) will be larger at smaller pixel sizes for a given \(m\) value. With \(V(m)\) larger, the signal-to-noise ratio will be enhanced, leading to more accurate inversion. Fine resolution will also provide more pixels, thus further increasing accuracies by increasing sample size. Accuracy should be further enhanced by the greater level of quantization of the SPOT data. As the result, the least-square error of fitting projected and generated \(m\) distributions to SPOT data will be much smaller than that of MSS data, and hence the estimated parameters of size and density will have smaller associated confidence intervals. Yet another advantage of the 10-m resolution is that we may safely assume the Poisson spacing model [35], and therefore \(C_d\) will be unity.

In light of these considerations, it was surprising to note that using the 80-m MSS data in inversion yielded estimates of size, height, spacing, and apex angle that were fairly close to the values obtained by field measurement (Table II). We view this result with caution, however, since it relies on a value for \(C_d\) that is obtained by projecting a linear regression well beyond the points used to fit the line.
D. Covariance Between Size and Spacing

In our earlier work with the nonoverlapping variance-dependent model [25], we noted that the assumption of independence between height and spacing did not hold for MSS pixels—rather, stands of large trees tended to be less dense than stands of smaller trees. Thus we used an empirical correction for this covariance in our inversion procedure that was calibrated by measurements of tree size and spacing in MSS-sized pixels using air photos. It is possible to go beyond a simple empirical correction and model the covariance explicitly under the assumption that tree counts in pixels are distributed as either normal or Poisson functions. Although space limitations preclude presenting a full derivation here, it is possible to show that

$$\text{cov}(n, R^2) = -C_d E(r^2)$$

under the assumption that the area of space nearest to any tree is directly proportional to its size (as $R^2$). Although the field data we collected in the red fir and mixed conifer sites which could be used to verify this relationship are limited, they suggest that the covariance is much smaller than (19) indicates for small pixel sizes (12-m or less). Thus we have ignored the effects of this covariance in the derivation of our model. At larger pixel sizes, and for sparse forest stands, it may be important to correct for this effect explicitly.

E. Apex Angle

The mathematical derivations of the geometric-optical model assume that the apex angle of the cones is a constant. Field data, however, show that measured apex angles may have a coefficient of variation as large as 0.585. When we invert the model through the least squares fitting procedure, we partly relax this assumption by treating $\alpha$ as a variable, projecting the two-dimensional tasseled cap pattern into a one-dimensional $m$ distribution and obtaining reasonably accurate estimates of $N$ and $R^2$. At the same time, we obtain a projected $\alpha$ distribution; if this distribution agrees well with the field measurements, then the tasseled cap brightness pattern of the coniferous forest is quantitatively and accurately explained by our geometric-optical model.

The variation in apex angle, however, raises the problem of the meaning of the average apex angle as an aggregation variable for the pixel or timber stand. If $\alpha$ is independent of $r$, we can take the average of $\alpha$ as parameterizing the form of the trees. The strong negative correlation between $\alpha$ and $r$, however, means that the simple average of $\alpha$ has little physical meaning. Under our assumptions, even if the $n$ cones within a pixel have different $\alpha$ and $r^2$ values, the reflectance of the pixel is still modeled as if it can be approximated by $n$ cones that have the same mean size $R^2$ and the same apex angle $\alpha$. Therefore, an obvious meaningful definition of the "average" $\alpha$ for a pixel is an equivalent angle such that the fixed apex angle model approximately holds. It is difficult to achieve a very accurate equivalence because the variation in form will affect all four signature components and influence the overlapping effect as well. Some approximations have to be made to simplify the problem. Since all the area terms of the reflectance components are of the form $r^2$ multiplied by a function of $\alpha$, the average apex angle we obtain from the projected $\alpha$ distribution will resemble a weighted average $E(r^2\alpha)/R^2$ for the pixel. This means that the larger cones will have greater weight in determining $\alpha$ for the pixel. The projected $\alpha$ distribution, we believe, should be understood in this sense.

If a similar weighted mean is calculated from the field data, the mean of the projected $\alpha$ distribution overestimates this value by about 13 percent. If we change the signatures as discussed in the section on sensitivity, the mean of the projected $\alpha$ distribution continues to overestimate the observed weighted mean by 2–45 percent. If we compare the mean of the projected $\alpha$ distribution to the simple average of $\alpha$, the simple average is underestimated by 25 percent, or 39–4 percent in sensitivity analysis. In short, our model produces an estimate of the mean apex angle $\alpha$, which lies between a simple average and an average weighted by $r^3$.

In our field work, we also noted that the coefficient of variation of the projected $\alpha$ distribution is about one-third of the coefficient of variation of measured crown apex angles. This underestimation results because the projected $\alpha$ values are averages over whole pixels. Our present field data, however, are insufficient to evaluate this aggregation effect quantitatively, since only four apex angles were measured at each subplot location.

F. Pixel Boundary Problem

Recall that in both the Monte Carlo simulation and the mathematical modeling of overlap between crowns and shadows, the pixel boundary problem was treated by replicating the portion of a crown or shadow that falls outside of the pixel on the inside of the pixel at the opposite boundary. In effect, the pixel overlaps onto itself. This simplification should underestimate pixel-to-pixel variance, and it is appropriate to examine its implications.

In the running of the Monte Carlo simulation, statistics were kept that recorded the proportions of the areas of $A_e$, $A_i$, and $A_f$ falling outside of the pixel boundary that were replicated within the pixel. At the 80-m pixel size, only a few percent of each area was affected; thus the influence on $K_e$, $K_i$, and $K_f$ will be negligible at MSS resolution. At 10-m resolution, however, these percentages increase to several tens, and thus the variances of the proportions increase significantly. And for large $m$ values, this cross-boundary variance can even be larger than the overlapping variance, since the latter is very small in this case (see Fig. 1). The field measurements also show that our expression for the variance with overlapping (15), which assumes infinite replication of the pixel, does not hold well at a pixel size smaller than about 12 m. This observation also indicates that cross-boundary variation will be a problem at this pixel size.

In order to evaluate the effects of this variation on our model, let us assume that $n$ trees are located within a pixel,
and they produce components \( A_c, A_s, \) and \( A_t \) as we previously modeled. Because of cross-boundary effects, parts of these areas, \( A_c', A_s', \) and \( A_t' \), will fall outside of the pixel, while \( A_c'', A_s'', \) and \( A_t'' \) from trees outside the pixel will fall inside. Let us consider the simplified cases.

1) \( A_c'/A_c'' \approx A_s'/A_s'' \approx A_t'/A_t'' \): In this case, the \( m' \) value inverted from (18) will deviate from \( mR^2 \) along the coverage trajectory. This value, however, can be thought of as a correct solution because pixel size is arbitrary and nothing prevents us from defining \( n \) as a real number, rather than an integer. Thus the deviation along the coverage trajectory does not affect the accuracy of the model.

2) \( A_c' + A_s' + A_t' \approx A_c'' + A_s'' + A_t'' \): In this case, the inversion of (18) will have errors basically only in \( \alpha \), since \( A_g \) does not change and it is the dominant factor determining the location of \( S \) along the coverage trajectory. Unless the value of \( \alpha \) obtained is far from the true value, the accuracy of the model will not be seriously affected, although the variance of \( \alpha \) will be inflated somewhat.

The actual situation will be a combination of these two cases, and will be more difficult to model exactly. However, it appears that unless the resolution is so fine that many pixels contain pure \( A_c, A_s, A_t \), or only fractions of crowns, the accuracy of this model will not be significantly degraded. Further Monte Carlo simulations conducted at 10-m resolution and using trees of the size encountered in the two study sites also show little error caused by the boundary problem. Thus our method of accommodating the boundary problem is probably reasonable for the types of forests and imagery we have encountered thus far in this research.

G. Further Application of the Geometric/Optical Model

From the numerical viewpoint, our model is driven by the variance encountered when the number of trees, their sizes, and their overlap varies from one pixel to the next. If this variance is not large, or contains a lot of noise, then inversion of the model will be inaccurate. As (15) shows, this can happen in two ways. First, if the pixel size is large, then \( a \), which is the ratio of the area of the object to the area of the pixel, will become smaller, in turn reducing \( V(A_g) \). Second, if the number of objects \( N \) is large, then the kernel of (14) will become smaller since \( q \) is a positive function between 0 and 1. In other words, large pixels will have low variance because of averaging, and densely covered pixels will have low variance since nearly all the background is obscured.

In our work to date, we have applied our model to pixels with sizes equal to or smaller than 80 m on a side and to canopy coverages of \( M < 0.46 \), which gives a cover of about 75 percent (red fir site). Even in the case of canopy covers greater than 75 percent, there may still be a sufficiently ample number of pixels from the low end of the \( m \) distribution to invert the model through the least-squares procedure (e.g., Fig. 16). Thus the procedure should work with even denser stands provided that the trees are not too small.

As to the problem of large pixel size, one helpful approach will be to estimate or calibrate the noise variance contained in \( V(m) \); since it will be independent and additive, it can be separated from the true variance unless \( V(m) \) is so small that it lies below the quantization level of the remotely sensed signal. This approach, however, is not likely to be needed, since we have already demonstrated sufficient variance at 80 m with 75-percent canopy cover. Given the abundance of Landsat MSS data at 80-m resolution, and the prospect of 7-band 30-m data from TM, it seems likely that the model will be widely applicable.

Another advantage of our approach is that it can be used to identify pixels that do not fit the model for the stand as a whole. If the individual value of \( \alpha \) or \( M \) obtained for a pixel departs greatly from modeled values, it can be classified as nonconifer and excluded from parametric estimation. In this way, the present requirement for predelineating a timber stand may be relaxed somewhat.

X. Conclusion

Our geometric-optical canopy model demonstrates that regarding a remotely sensed scene as consisting of three-dimensional objects casting shadows on a background can be highly productive. Our simulations show that the three-dimensional geometry alone can go a long way toward explaining the bidirectional reflectance distribution function of forest canopies, thus emphasizing the importance of shape, form, and shadowing of objects in influencing images of real scenes.

In addition, explicit modeling of shadowing geometry in the presence of overlapping shadows and objects has allowed the development of an invertible model that yields the average size and spacing of objects, even when the objects are considerably smaller than the pixels. The inversion requires specifying and modeling the interpixel variance within a uniform area, and in this sense our model is probably the first canopy model to utilize the fact that pixels from a uniform area may be regarded as different realizations of random processes containing information about the processes themselves. Our model is therefore spatial in the sense that the realizations are drawn from a spatially connected field, and requires an image if it is to be inverted. Thus the geometric-optical model differs from nearly all other plant canopy models in that it is designed specifically to explain, utilize, and exploit the spatial variation inherent in digital imagery.
Although our geometric-optical model exploits spatial variation, it does not truly utilize the spatial dimension in that the absolute or relative position of each measurement is not required. It therefore does not make full use of all the information inherent in a digital image. A planned future thrust of our modeling activity is to use spatial position directly in model formulation. When resolution cell size is small and the objects are large (as with SPOT or TM imagery), adjacent pixels will influence one another because objects are discrete entities that extend across pixel boundaries. In explicitly modeling this effect, the cross-boundary variance is therefore not a problem to be overcome, but a source of information that can be used to advantage to determine the size and spacing of the objects more accurately. Such "third-generation" models will truly exploit the full potential of remote sensing to reveal the spatial parameters and processes that shape the Earth's landscape.

REFERENCES


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